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# On the Representation Theorem for Quantum Logic

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*Abstract*

The main assumption of Varadarajan's version of Piron's representation theorem for quantum logic, stating that the lattice under any finite element of the logic is a geometry of finite rank, is eliminated by means of more plausible assumptions, concerning the property of symmetry of the transition probability between pure states. It is also proved, that the quantum logic with symmetric transition probability is irreducible iff it is completely irreducible.

## 1. *Introduction*

The logical approach was created to solve the basic problem of quantum theories: are Hilbert spaces an appropriate tool for the description of quantum phenomena? This question is currently the main point of investigations in the quantum logic area. The positive solution of this problem is given by formulation of a list of plausible physical assumptions, which result in the isomorphism of quantum logic and the logic associated to either a real, complex or quaternionic Hilbert space. All proposed sets of axioms of this type may be criticised because of the subjective meaning of the 'physical plausibility'. In the authoress's opinion, the most satisfactory result is Varadarajan's version of Piron's representation theorem (Varadarajan, 1968; pp. 176–184; Piron, 1964). However, Varadarajan's assumption that the lattice of all elements contained in a finite element of the logic is a geometry of finite rank, is particularly unfortunate, as Gudder (1970) also has noticed. The axioms, proposed by Gudder in place of that, seem to be also not entirely plausible, though formulated in intuitively clearer terms.

In this paper, the assumption of Varadarajan already mentioned is shown to be a consequence of the more physical postulate, requiring that the transition probability $p(\alpha,\beta)$ between pure states $\alpha,\beta$ is symmetric: $p(\alpha,\beta) = p(\beta,\alpha)$. This property of transition probability is commonly accepted and considered as unquestionable.

## 2. *Postulates*

The set $\mathscr{L}$ of all elementary (yes–no) measurements, called the logic, and the set $\mathscr{S}$, called the set of states, are the main objects of studies in the quantum logic approach. The following postulates express the features of $\mathscr{L}$ and $\mathscr{S}$ usually assumed.

*Postulate* 1. Logic $\mathscr{L}$ is an orthomodular, complete, atomic ortho-lattice.

*Postulate* 2. Set of states $\mathscr{S}$ is a $\sigma$-convex subset of the set of all probability measures on $\mathscr{L}$, such that $a \leqslant b$, $a, b \in \mathscr{L}$ iff $\alpha(a) \leqslant \alpha(b)$ for all $\alpha \in \mathscr{S}$.

These standard and often-discussed assumptions of the quantum logic approach are in agreement with physical intuition. The set of states, as well as Postulate 2, is omitted in the afore-mentioned considerations of Piron and Varadarajan. This restriction seems to be superflous because the notion of state has a good experimental meaning in quantum physics.

The next postulate concerns a relation between set $\mathscr{A}$ of all atoms of $\mathscr{L}$ and set $\mathscr{P}$ of all pure states.

*Postulate* 3. (i) Set $\mathscr{P}$ is non-empty.

(ii) If $\alpha \in \mathscr{P}$, then there exists an atom $a$ such that $\alpha(b) = 1 \Rightarrow a \leqslant b$ for all $b \in \mathscr{L}$.

(iii) If $a \in \mathscr{A}$, then there exists one and only one pure state $\alpha$ such that $\alpha(b) = 1 \Rightarrow a \leqslant b$ for all $b \in \mathscr{L}$.

If $\alpha$ is a pure state, then the atom $a$ such that $\alpha(b) = 1$ implies $a \leqslant b$ for all $b \in \mathscr{L}$, is called the carrier of $\alpha$ and denoted $\mathrm{car}(\alpha)$. The phenomenological meaning of Postulate 3 is evident. The carrier of pure state $\alpha$ represents a laboratory device, answering the experimental question: is the system in state $\alpha$? Postulate 3 asserts, that it is possible to identify the pure state by means of one elementary measurement.

Let us observe that the number $p(\alpha, \beta) = \beta(\mathrm{car}(\alpha)), \alpha, \beta \in \mathscr{P}$, has the meaning of the transition probability from pure state $\alpha$ to pure state $\beta$. This number is well defined owing to Postulate 3. By Postulate 3 one can also define the transition probability between atoms as equal to the transition probability between corresponding pure states.

It is easy to see that:

(i) $0 \leqslant p(\alpha, \beta) \leqslant 1$ for all $\alpha, \beta \in \mathscr{P}$,
(ii) $p(\alpha, \alpha) = 1$ for all $\alpha \in \mathscr{P}$,
(iii) $p(\alpha, \beta) = 0 \Leftrightarrow \mathrm{car}(\alpha) \perp \mathrm{car}(\beta)$ for all $\alpha, \beta \in \mathscr{P}$,

with the orthogonality relation $a \perp b$ defined as $a \leqslant b'$.

However, the assumed postulates do not imply the property of symmetry for transition probability. We assume:

*Postulate* 4. $p(\alpha, \beta) = p(\beta, \alpha)$ for every pair $\alpha, \beta \in \mathscr{P}$.

This natural property makes possible the elimination of the main hypothesis of Varadarajan's theorem concerning the modularity of the lattice under any finite element of $\mathscr{L}$. Nevertheless, we shall make use of the second assumption of Varadarajan:

*Postulate* 5. If $b \neq o, e$ (the least and the greatest elements of $\mathscr{L}$ respectively) and $a \in \mathscr{A}$, then there exists atoms $a_1$, $a_2$ such that $a_1 \leqslant b'$, $a_2 \leqslant b$ and $a \leqslant a_1 \vee a_2$.

This assumption is related to the quantum-mechanical 'projection postulate'.

The last axiom is of rather a formal nature:

*Postulate* 6. $\mathscr{L}$ is irreducible.

If Postulate 6 does not hold, i.e. if there are super-selection rules imposed on the system, then any irreducible 'sector' of the whole logic may be taken as $\mathscr{L}$. Thus the last postulate is not restrictive.

For a further discussion of Postulates 1 to 4, the reader is referred to the paper of Bugajska & Bugajski (1973).

## 3. *Theorems*

The present section is devoted to the proof that the mentioned hypothesis of Varadarajan's theorem is a consequence of our Postulates 1 to 6. The proof is based on a series of lemmas.

Logic $\mathscr{L}$, by Postulate 1, is atomic, hence set $a_0{}^{\Delta} = \{b \in \mathscr{L} | b \leqslant a_0\}$, $a_0 \in \mathscr{L}$, contains at least one atom. We denote the set $a_0{}^{\Delta} \cap \mathscr{A}$ of all atoms contained in $a_0$ by $\mathscr{A}(a_0)$. One can find in $\mathscr{A}(a_0)$ subsets of pairwise orthogonal atoms. Amongst these subsets there exists a maximal one, by Zorn's lemma.

*Definition* 1. An orthobasis of $a_0$ is a maximal subset of pairwise orthogonal atoms contained in $a_0$.

Obviously, if $\{a_i, i \in J\}$, $J$—some indexing set, is an orthobasis of $a_0$, then $a_0 = \bigvee_{i \in J} a_i$ (by Postulate 1).

*Lemma* 1. If the countable orthobasis $\{a_n, n \in N\}$ of $a_0$ exists, then any other orthobasis of $a_0$ is countable.

*Proof.* Let $\{b_i, i \in J\}$ be an orthobasis of $a_0$. One can prove, that if $a$ is an atom, then $p(a, b_i) \neq 0$ only for some countable set of indices $J(a) = \{i_1, i_2, \ldots\} \subset J$. Thus set $J(a_1) \cup J(a_2) \cup J(a_3) \cup \ldots$ is countable. If $i \in J \backslash (J(a_1) \cup J(a_2) \cup \ldots)$ then $b_i \perp a_n$ for all $n \in N$. Thus $b_i \perp a_1 \vee a_2 \vee \ldots = a_0$ and $J = J(a_1) \cup J(a_2) \cup \ldots$.

*Lemma* 2. If the orthobasis $\{a_1, a_2, \ldots a_k\}$ of $a_0$, composed of $k$ elements, exists, then any other orthobasis of $a_0$ is composed of $k$ elements.

*Proof.* We adopt the proof of the analogous statement given by Mielnik (1968). Let $\{b_n, n \in N\}$ be an orthobasis of $a_0$. If $a$ is an atom of $a_0$, then $\sum_{n \in N} p(a, b_n) = 1$. Thus $\sum_{m=1}^{k} \sum_{n \in N} p(a_m, b_n) = k$ and byint erchanging the order of summation we obtain $\sum_{n \in N} 1 = k$.

Lemma 2 enables us to introduce the notion of dimension of an element of $\mathscr{L}$:

*Definition* 2. If $a$ possesses a finite orthobasis, composed of $k$ elements, then $a$ will be called a finite element of $\mathscr{L}$ and the number $k$ the dimension of $a, k = \dim(a)$ in symbols.

We shall prove such defined $\dim(a)$ to be the dimension function, i.e. to have the properties:

(i) $\dim(o) = 0, \dim(a) \geqslant 0$,
(ii) $a \leqslant b$ and $a \neq b \Rightarrow \dim(a) < \dim(b)$,
(iii) $\dim(a \vee b) + \dim(a \wedge b) = \dim(a) + \dim(b)$,

for every finite $a, b \in \mathscr{L}$.
The first property is obvious, as well as the second (by Postulate 1). Before deriving property (iii), we shall prove

*Lemma* 3. If $b$ is a finite element of $\mathscr{L}$, and $a$ is an atom, $a \notin \mathscr{A}(b)$, then $\dim(b \vee a) = \dim(b) + 1$.

*Proof.* It follows from Postulate 5, that there are atoms $a_1$, $a_2$ such that $a_1 \leqslant b'$, $a_2 \leqslant b$, $a \leqslant a_1 \vee a_2$. Hence $b \vee a \leqslant b \vee a_1 \vee a_2 = b \vee a_1$. On the other hand, there exists by Postulate 1, an atom $b_1$ such that $b_1 \perp b$ and $b \vee b_1 \leqslant b \vee a$. Consequently, $b \vee b_1 \leqslant b \vee a \leqslant b \vee a_1$. Thus, by Postulate 1 and Lemma 2, $b \vee b_1 = b \vee a_1$. This proves that $b \vee a = b \vee a_1$ and $\dim(b \vee a) = \dim(b) + 1$.
The two following corollaries are implied by the above lemma:

*Corollary* 1. If $a_1 a_2, \ldots a_k$ are atoms, and $a_0 = a_1 \vee a_2 \vee \ldots \vee a_k$ then $a_0$ is finite and $\dim(a_0) \leqslant k$.

*Corollary* 2. If $a,b$-finite elements of $\mathscr{L}$, then $a \vee b$ is finite and $\dim(a \vee b) + \dim(a \wedge b) \leqslant \dim(a) + \dim(b)$.
Now we are able to prove property (iii) of the function $\dim(a)$:

*Lemma* 4. If $a,b$-finite elements of $\mathscr{L}$, then $\dim(a \vee b) + \dim(a \wedge b) = \dim(a) + \dim(b)$.

*Proof.* Let us denote $a \vee b$ by $a_0$ and $\dim(a_0)$ by $k$. If $c \leqslant a_0$, then we define relative orthocomplement $c^0$ of $c$: $c^0 = c' \wedge a_0$. Obviously, $\dim(c^0) = k - \dim(c)$. It is easy to see that $(c_1 \vee c_2)^0 = c_1{}^0 \wedge c_2{}^0$ and $(c_1 \wedge c_2)^0 = c_1{}^0 \vee c_2{}^0$ for any $c_1, c_2 \leqslant a_0$. Corollary 2 states that

$$\dim(a^0 \vee b^0) + \dim(a^0 \wedge b^0) \leqslant \dim(a^0) + \dim(b^0).$$

Hence

$$k - \dim(a \wedge b) + k - \dim(a \vee b) \leqslant k - \dim(a) + k - \dim(b)$$

and

$$\dim(a \vee b) + \dim(a \wedge b) \geqslant \dim(a) + \dim(b).$$

Thus $\dim(b)$ is a dimension function on the lattice $a^\Delta$ for any finite element $a$ of $\mathscr{L}$. It is proved (Varadarajan, 1968, p. 22) that if a dimension function is defined on some lattice, then the lattice in question is modular. Consequently, the following theorem holds:

*Theorem* 1. Let $\mathscr{L}$ and $\mathscr{S}$ satisfy Postulates 1 to 5. Then $a^{\Delta}$ is a modular lattice for every finite $a \in \mathscr{L}$.

However, Varadarajan's hypothesis states that $a^{\Delta}$ is a geometry of finite rank, provided $a$ is finite. Hence we must prove that if $a$ is finite then $a^{\Delta}$ is irreducible, i.e. that the logic $\mathscr{L}$ is completely irreducible (in terms of Gudder, 1970).

We shall prove the equivalence of the complete irreducibility and irreducibility of $\mathscr{L}$. The proof is based on the following lemma:

*Lemma* 5. If $a_1$, $a_2$, $b$ are three distinct atoms and $\mathscr{A}(a_1 \vee a_2) = \{a_1, a_2\}$, then

$$\mathscr{A}(a_1 \vee a_2 \vee b) = \{a_1\} \cup \mathscr{A}(a_2 \vee b)$$

or

$$\mathscr{A}(a_1 \vee a_2 \vee b) = \{a_2\} \cup \mathscr{A}(a_1 \vee b)$$

*Proof.* If $\mathscr{A}(a_1 \vee a_2) = \{a_1, a_2\}$, then $a_1 \perp a_2$. Element $a_1 \vee a_2 \vee b$ of $\mathscr{L}$ is three-dimensional, hence there exists atom $c$ such that $c \perp a_1, a_2$ and $a_1 \vee a_2 \vee b = a_1 \vee a_2 \vee c$. Let $c_1 \in \mathscr{A}(a_1 \vee a_2 \vee c)$ and $c_1 \neq a_1, a_2, c$, then

$$\dim((c_1 \vee c) \wedge (a_1 \vee a_2)) = \dim(c_1 \vee c) + \dim(a_1 \vee a_2) - \dim((c_1 \vee c) \vee (a_1 \vee a_2)) = 1$$

Hence $(c_1 \vee c) \wedge (a_1 \vee a_2) = a_1$ or $a_2$ and $c_1 \in \mathscr{A}(a_1 \vee c)$ or $\in \mathscr{A}(a_2 \vee c)$. Thus $\mathscr{A}(a_1 \vee a_2 \vee c) = \mathscr{A}(a_1 \vee c) \cup \mathscr{A}(a_2 \vee c)$.

Let $c_1 \in \mathscr{A}(a_1 \vee c)$, $c_2 \in \mathscr{A}(a_2 \vee c)$ and $c_1, c_2 \neq a_1, a_2, c$. Then

$$\dim((c_1 \vee c_2) \wedge (a_1 \vee a_2)) = 1 \quad \text{and} \quad c_1 = a_1 \quad \text{or} \quad c_2 = a_2,$$

contrary to our assumption. Consequently:

$$\mathscr{A}(a_1 \vee a_2 \vee c) = \{a_1\} \cup \mathscr{A}(a_2 \vee c) \quad \text{or} \quad = \{a_2\} \cup \mathscr{A}(a_1 \vee c).$$

It is implied that $b \in \mathscr{A}(a_1 \vee c)$ or $\in \mathscr{A}(a_2 \vee c)$ and $\mathscr{A}(a_1 \vee a_2 \vee b) = \{a_1\} \cup \mathscr{A}(a_2 \vee b)$ or $= \{a_2\} \cup \mathscr{A}(a_1 \vee b)$.

An obvious consequence of the lemma is

*Corollary* 3. If $a_1$, $a_2$, $b$ are distinct atoms, and $\mathscr{A}(a_1 \vee a_2) = \{a_1, a_2\}$ then:

(i) $b \perp a_1$ or $b \perp a_2$ or $b \perp a_1, a_2$,
(ii) $\mathscr{A}(b \vee a_1) = \{b, a_1\}$ or $\mathscr{A}(b \vee a_2) = \{b, a_2\}$.

Lemma 5 enables us to prove

*Theorem* 2. Let $\mathscr{L}$ and $\mathscr{S}$ satisfy Postulates 1 to 5. Then $\mathscr{L}$ is irreducible iff $\mathscr{L}$ is completely irreducible.

*Proof.* The first assertion of the theorem: $\mathscr{L}$ is completely irreducible $\Rightarrow \mathscr{L}$ is irreducible, is obvious. Let us prove the second one, assuming $\mathscr{L}$ not to be completely irreducible, i.e. assuming that there exists a finite element $a$ of $\mathscr{L}$, such that the modular lattice $a^{\Delta}$ is reducible. That causes the existence of two atoms $a_1$, $a_2$ such that $\mathscr{A}(a_1 \vee a_2) = \{a_1, a_2\}$. Let $\mathscr{A}_1 = \mathscr{A}\backslash\mathscr{A}(a_1')$, $\mathscr{A}_2 = \mathscr{A}\backslash\mathscr{A}(a_2')$. Set $\mathscr{A}_i$ contains $a_i$ and all atoms noncompatible with $a_i$ $(i = 1, 2)$. If $c \in \mathscr{A}_2$, then $\mathscr{A}(c \vee a_1) = \{c, a_1\}$ and $c \perp a_1$, by Corollary 3. Similarly, if $c \in \mathscr{A}_1$, then $\mathscr{A}(c \vee a_2) = \{c, a_2\}$ and $c \perp a_2$. Applying Lemma 4 to $c_1 \in \mathscr{A}_1$, $c_2 \in \mathscr{A}_2$ and $a_1$ we obtain $c_1 \perp c_2$, i.e. $\mathscr{A}_1 \perp \mathscr{A}_2$. In an analogous manner one can prove that if $d \in \mathscr{A}_3 = \mathscr{A}\backslash(\mathscr{A}_1 \cup \mathscr{A}_2)$, then $d \perp c_1$ and $d \perp c_2$ for any $c_1 \in \mathscr{A}_1$ and $c_2 \in \mathscr{A}_2$. Thus the set of atoms $\mathscr{A}$ is divided into three disjoint and mutually orthogonal subsets: $\mathscr{A}_1, \mathscr{A}_2, \mathscr{A}_3$. The elements $\overline{a_1} = \bigvee_{b \in \mathscr{A}_1} b$, $\overline{a_2} = \bigvee_{b \in \mathscr{A}_2} b$, $\overline{a_3} = \bigvee_{b \in \mathscr{A}_3} b$ are compatible with all elements of $\mathscr{L}$. Consequently, $\mathscr{L}$ is reducible. This proves the desired result: $\mathscr{L}$ is irreducible $\Rightarrow \mathscr{L}$ is completely irreducible.

The above series of lemmas and theorems may be summarised as

*Theorem* 3. If $\mathscr{L}$ and $\mathscr{S}$ satisfy Postulates 1 to 6, then $a^{\Delta}$ is a geometry of finite rank for every finite $a \in \mathscr{L}$.

## 4. *Conclusions*

The result of the previous section enables one to give a more plausible hypotheses than the original ones for Varadarajan's representation theorem. Thus Varadarajan's (1968, p. 179) theorem takes the form:

*Theorem* 4. Let $\mathscr{L}$ and $\mathscr{S}$ satisfy Postulates 1 to 6 of Section 2 and let there exist at least one $a \in \mathscr{L}$ such that $\dim(a) \geqslant 4$. Then there exists a division ring **D**, an involutive anti-automorphism $\theta$ of **D**, a vector space $V$ over **D**, and a definite symmetric $\theta$-bilinear form $\langle \cdot\, , \cdot \rangle$ on $V \times V$ such that $(V, \langle \cdot\, , \cdot \rangle)$ is Hilbertian and $\mathscr{L}$ is isomorphic to the logic of all $\langle \cdot\, , \cdot \rangle$-closed linear manifolds of $V$.

The assumed postulates seem to be satisfactory (with the possible exception of Postulate 5) from the phenomenological point of view. It is very probable that every quantum system with symmetric transition probability may be described in terms of some Hilbertian vector space. Thus the symmetry of transition probability appears to be a criterion for the existence of the 'vector space representation' of quantum logic.

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